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DYNAMIC ANALYSIS OF RECTANGULAR PLATES WITH STEPPED THICKNESS SUBJECTED TO MOVING LOADS INCLUDING ADDITIONAL MASS

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This paper presents a simplified analytical method for rectangular plates with arbitrarily- and eccentrically-stepped thickness, such as building slabs, subjected to moving loads, including the effect of the additional mass. The discontinuous variation of the bending stiffness and mass of the plate due to the variation of the thickness can be expressed continuously by means of a characteristic function, which is defined as a Dirac function existing continuously in a prescribed region, as proposed by the author's previous work. Since the bending stiffness used here is given exactly by the actual bending stiffness at each point, a modification of the bending stiffness used in an equivalent plate analogy is unnecessary. It is clarified from numerical computations that the effect of the additional mass due to moving loads is significant for heavy-weight additional mass but is negligible for usual additional mass with 65 kg. Then, approximate but accurate solutions for a current plate subjected to stationary and moving loads are proposed by excluding the effect of additional mass. The numerical results obtained from the proposed theory for simply-supported and clamped plates with stepped thickness, excluding the effect of additional mass, show good agreement with results obtained from the finite element method using FEM code NASTRAN.

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1. INTRODUCTION

Recently, a building slab with a relatively thin stepped thickness has been used frequently to decrease cost and to simplify construction. There is relatively little research work, dealing with such a plate of stepped thickness. Juarez [1], Chopra [2], Laura and Filipich [3], Filipich *et al*. [4], Sakata [5], Cortinez and Laura [6], and Cheung and Kong [7] discussed about a plate with stepped thickness. In building slabs, however, the upper surface of slabs is flat over the whole surface, and the variation of the thickness is restricted on the lower surface. In such a plate with eccentrically-stepped thickness, the variation of mass and rigidity are discontinuous. Takabatake *et al*. [8] presented a simplified analysis for the static and dynamic problems of a rectangular plate with arbitrarily- and eccentrically-stepped thickness. The discontinuous variation of the bending stiffness and mass of such a plate due to arbitrarily-stepped thickness is expressed rationally as a continuous function by means of a characteristic function proposed by Takabatake [9–11] and Takabatake *et al*. [12–14]. The characteristic function is defined as a Dirac function existing continuously in a prescribed region.

On the other hand, recently human response to vibration in building slabs has become significant in design. The practical dynamic response must be calculated for moving loads. The effect of the additional mass due to moving loads on slabs is unknown and is lacking. The use of FEM code for moving loads is very costly and troublesome. Therefore, the

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analytical check to human response in practical design is judged from the response of slabs, subjected to a concentrated load of unmoving foot impact at the midspan by means of FEM. A simplified analysis being practical and usable for building slabs, subjected to stationary and moving loads, is also demanded, so that the problems due to the vibration of building slabs is removed in the preliminary stages and final stages of the design. Although the use of FEM for building slabs is effective, it needs a computer of great capacity and is costly and time-consuming for computation. Furthermore, it cannot solve the dynamic problem including the effect of additional mass due to moving loads.

The purpose of this paper is to demonstrate a simplified analysis of an isotropic plate with relatively thin eccentrically-stepped thickness, subjected to moving loads along with the effect of additional mass. First, the effect of additional mass due to moving loads is presented by modifying the governing equation proposed by Takabatake *et al*. [8] and is clarified from numerical computations. Second, when the effect of additional mass due to moving loads is negligible, approximate solutions as closed-form are presented. The accuracy of the proposed solutions is established from numerical computation for simply-supported and clamped plates.

2. GOVERNING EQUATIONS OF A PLATE WITH STEPPED THICKNESS INCLUDING THE EFFECT OF MOVING ADDITIONAL MASS

Consider a rectangular plate with arbitrarily- and eccentrically-stepped thickness, as shown in Figure 1. The Cartesian co-ordinate system x, y, z is employed. Ridgelines of each eccentrically-stepped thickness are assumed to be parallel to the x- or y-axis. The midpoint, width, and varied thickness of the ith stepped thickness being parallel to the y-axis are indicated by x_i, b_{xi}, and h_{xi}, respectively, in which h_{xi} is measured from the lower surface of the reference slab excluding stepped thickness. Similarly for the jth stepped thickness being parallel to the x-axis they are given by y_j, b_{yj}, and h_{yj}. In a part where the ith and jth stepped thickness being parallel to the x- and y-axes, respectively, cross each other the maximum value of their stepped thickness h_{xi} and h_{yj} is indicated with $h_{xyi,j}$.

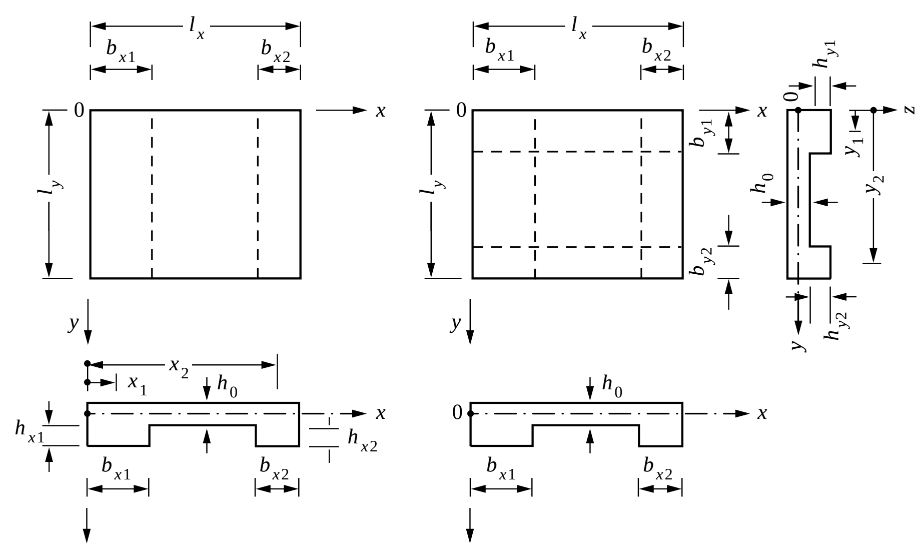

Figure 1. Co-ordinates of a rectangular plate with arbitrarily-stepped thickness.

The variation of a slab's thickness in most building slabs is relatively small. Therefore, for the sake of simplicity, the neutral surface of a current plate is assumed to be located on a surface that bisects the thickness at each point; and to vary discontinuously at the boundary line between eccentrically-stepped thickness and reference thickness. This engineering assumption, as shown in previous work [8], makes the formulation of the current plate simple and is proven to be practically effective within the range of $h_{xi}/h_0 = 1.0$ and $h_{yj}/h_0 = 1.0$, in which h_0 is the thickness of the reference slab.

Then, consider the bending problem of isotropic rectangular plates in small deformations based on the validity of the Kirchhoff–Love hypotheses. Using the above engineering assumption to the neutral surface, the flexural rigidity and mass of current plates with stepped thickness, which are functions of x and y, are expressed by $D_0\, d(x, y)$ and $m_0\, \alpha_h(x, y)$, respectively, in which D_0 and m_0 are the flexural rigidity and mass of a reference plate neglecting stepped thickness, as given by $D_0 = Eh_0^3/[12(1 - \nu^2)]$ and $m_0 = \rho h_0$, respectively. Here E and ν are Young's modulus and Poisson's ratio, respectively, and ρ is the mass density of the plate. On the other hand, the coefficients of flexural rigidity and mass, $d(x, y)$ and $\alpha_h(x, y)$, respectively, are defined as

$$d(x, y) = 1 + \alpha_{xi} D(x - x_i) + \alpha_{yj} D(y - y_j) + \alpha_{xyi,j} D(x - x_i) D(y - y_j), \tag{1}$$

$$\alpha_h(x, y) = 1 + \left(\frac{h_{xi}}{h_0}\right) D(x - x_i) + \left(\frac{h_{yj}}{h_0}\right) D(y - y_j) + \left(\frac{h_{xyi,j} - h_{xi} - h_{yj}}{h_0}\right) D(x - x_i) D(y - y_j), \tag{2}$$

in which constants α_{xi}, α_{yj}, $\alpha_{xyi,j}$ are given by

$$\alpha_{xi} = 3\frac{h_{xi}}{h_0} + 3\left(\frac{h_{xi}}{h_0}\right)^2 + \left(\frac{h_{xi}}{h_0}\right)^3, \tag{3}$$

$$\alpha_{yj} = 3\frac{h_{yj}}{h_0} + 3\left(\frac{h_{yj}}{h_0}\right)^2 + \left(\frac{h_{yj}}{h_0}\right)^3, \tag{4}$$

$$\alpha_{xyi,j} = 3\frac{h_{xyi,j}}{h_0} + 3\left(\frac{h_{xyi,j}}{h_0}\right)^2 + \left(\frac{h_{xyi,j}}{h_0}\right)^3 - \alpha_{xi} - \alpha_{yj}, \tag{5}$$

and $D(x - x_i)$ and $D(y - y_j)$ are characteristic functions of the extended Dirac function. The function $D(x - x_i)$ is defined as a function where the Dirac function $\delta(x - \xi)$ exists continuously in the x direction through the ith stepped thickness, namely the region from $x_i - b_{xi}/2$ to $x_i + b_{xi}/2$, in which ξ can take values continuously from $x_i - b_{xi}/2$ to $x_i + b_{xi}2$. Similarly, the function $D(y - y_j)$ is defined as a function where the Dirac function $\delta(y - \eta)$ exists continuously in the y direction through the jth stepped thickness, in which η can take values continuously from $y_j - b_{yj}/2$ to $y_j + b_{yj}/2$. The characteristic functions $D(x - x_i)$ and $D(y - y_j)$ proposed here are significant only in prescribed regions, but meaningless in other regions. Namely, their functions have the same effect as the Dirac function defined at a point inside the prescribed regions. Furthermore, the greatest merit of these functions is to make formulation and integral including discontinuous quantity, as shown in equations (17) and (18), more simple.

Thus, the governing equation of a plate with stepped thickness, including the effect of additional mass due to moving loads, is obtained by modifying the result of Takabatake *et al*. [8] as follows:

$$\frac{m_0\alpha_h\ddot{w}}{D_0}+\frac{\dot{\bar{m}}\dot{w}}{D_0}+\frac{\bar{m}\ddot{w}}{D_0}+\frac{c\dot{w}}{D_0}-\frac{p}{D_0}+(\mathrm{d}w_{,xx})_{,xx}+(\mathrm{d}w_{,yy})_{,yy}$$
$$+v(\mathrm{d}w_{,xx})_{,yy}+v(\mathrm{d}w_{,yy})_{,xx}+2(1-v)(\mathrm{d}w_{,xy})_{,xy}=0 \tag{6}$$

together with the associated boundary conditions

$$w=\bar{w} \qquad \text{or} \qquad (D_0\mathrm{d}w_{,xx})_{,x}+v(D_0\mathrm{d}w_{,yy})_{,x}+2(1-v)(D_0\mathrm{d}w_{,xy})_{,y}+v_x=0, \tag{7}$$

$$w_{,x}=\bar{w}_{,x} \qquad \text{or} \qquad D_0\mathrm{d}(w_{,xx}+vw_{,yy})+m_x=0, \tag{8}$$

at $x=0$ and l_x; and

$$w=0 \qquad \text{or} \qquad -(1-v)D_0\,\mathrm{d}w_{,xy}=m_{xy} \tag{9}$$

at the corners, in which $\bar{m}$ is the additional mass due to moving loads, w is the deflection on the natural surface of the current plate, $\bar{w}$ and $\bar{w}_{,x}$ are displacement and rotation, respectively, prescribed from the geometrical boundary conditions, and m_x and v_x are moment and Kirchhoff's supplementary force, respectively, prescribed from the mechanical boundary conditions. The similar boundary conditions may be written at $y=0$ and l_y.

For uniform solid plates, since the characteristic functions $D(x-x_i)$ and $D(y-y_j)$ are meaningless, the coefficients $d(x,y)$ and $\alpha_h(x,y)$ become one and the governing equations are reduced to a general equation, including the effect of additional mass, for a uniform plate.

3. FORCED VIBRATION OF A PLATE WITH STEPPED THICKNESS

The general solution of equation (6) is assumed to be of the form

$$w(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}f_{xm}(x)f_{yn}(y)\Phi_{mn}(t), \tag{10}$$

in which $\Phi_{mn}(t)$ are unknown functions with respect to time t, and $f_{xm}(x)$ and $f_{yn}(y)$ are the natural functions satisfying the specified boundary conditions of the current plate. As stated before, the discontinuous variations of flexural rigidity and mass are expressed continuously by means of a characteristic function extended from the Dirac function. Therefore, the current analysis is unnecessary to prepare an imaginary cutting at the discontinuities of rigidity and mass. Hence, although the assumption for the neutral surface is used on the discontinuities, the continuous condition of the deflection at the discontinuities is satisfied from natural functions used.

Applying the Galerkin method to equation (6), Φ_{mn} are obtained, from Takabatake *et al*. [8], by solving the following equations:

$$\sum_{m=1}\sum_{n=1}K_{\bar{m}\bar{n}mn}[\ddot{\Phi}_{mn}(t)+2h_{mn}\omega_{mn}\dot{\Phi}(t)+\omega_{mn}^2\Phi_{mn}(t)]$$

$$+\sum_{m=1}\sum_{n=1}[K^{(1)}_{\bar{m}\bar{n}m}\ddot{\Phi}_{mn}(t)+K^{(2)}_{\bar{m}\bar{n}m}\dot{\Phi}_{mn}(t)]\frac{1}{m_0}=\frac{1}{m_0}Q_{\bar{m}\bar{n}}(t), \tag{11}$$

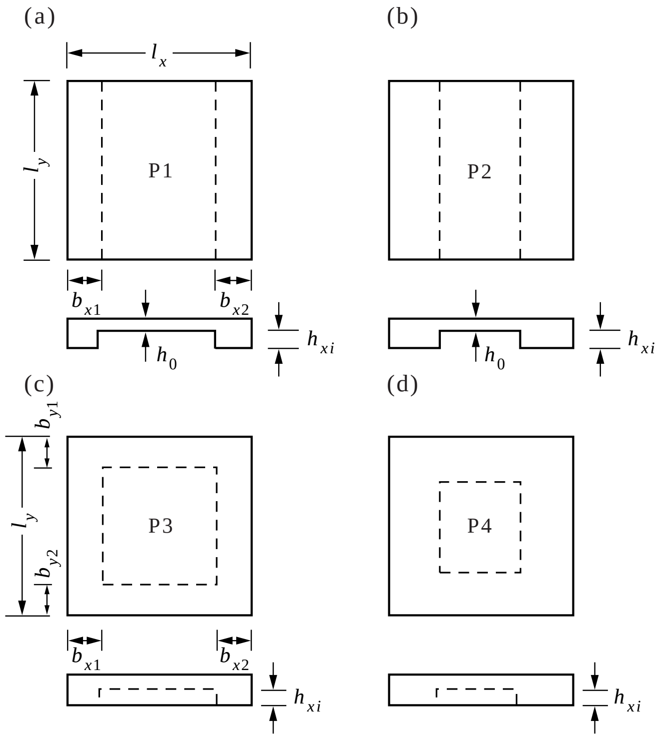

Figure 2. Isotropic rectangular plates with stepped thickness: (a) $b_{xi}/l_x = 0{\cdot}2$ ($i = 1, 2$); (b) $b_{xi}/l_x = 0{\cdot}3$ ($i = 1, 2$); (c) $b_{xi}/l_x = 0{\cdot}2$ ($i = 1, 2$), $b_{yj}/l_y = 0{\cdot}2$ ($j = 1, 2$); (d) $b_{xi}/l_x = 0{\cdot}3$ ($i = 1, 2$), $b_{yj}/l_y = 0{\cdot}3$ ($j = 1, 2$).

in which the notations $K_{\bar{m}\bar{n}mn}$, $K^{(1)}_{\bar{m}\bar{n}mn}$, $K^{(2)}_{\bar{m}\bar{n}mn}$ and $Q_{\bar{m}\bar{n}}(t)$ are defined as

$$K_{\bar{m}\bar{n}mn} = \int_0^{l_x}\int_0^{l_y} \alpha_h(x, y) f_{xm} f_{x\bar{m}} f_{yn} f_{y\bar{n}} \, \mathrm{d}x \, \mathrm{d}y, \tag{12}$$

$$K^{(1)}_{\bar{m}\bar{n}mn}(t) = \int_0^{l_x}\int_0^{l_y} \bar{m}(x, y, t) f_{xm} f_{x\bar{m}} f_{yn} f_{y\bar{n}} \, \mathrm{d}x \, \mathrm{d}y, \tag{13}$$

$$K^{(2)}_{\bar{m}\bar{n}mn}(t) = \int_0^{l_x}\int_0^{l_y} \dot{\bar{m}}(x, y, t) f_{xm} f_{x\bar{m}} f_{yn} f_{y\bar{n}} \, \mathrm{d}x \, \mathrm{d}y, \tag{14}$$

$$Q_{\bar{m}\bar{n}}(t) = \int_0^{l_x}\int_0^{l_y} p(x, y, t) f_{x\bar{m}} f_{y\bar{n}} \, \mathrm{d}x \, \mathrm{d}y, \tag{15}$$

and h_{mn} are damping constants. The coefficients $K^{(1)}_{\bar{m}\bar{n}mn}$ and $K^{(2)}_{\bar{m}\bar{n}mn}$ are variable with respect to time. Therefore, equation (11) can be solved by using the Wilson-θ method.

The explicit expression of equation (12) is written as

$$K_{\bar{m}\bar{n}mn} = \int_0^{l_x}\int_0^{l_y} f_{xm}f_{x\bar{m}}f_{yn}f_{y\bar{n}}\,\mathrm{d}x\,\mathrm{d}y + \sum_{i=1}\frac{h_{xi}}{h_0}\int_0^{l_x}\int_0^{l_y} D(x-x_i)f_{xm}f_{x\bar{m}}f_{yn}f_{y\bar{n}}\,\mathrm{d}x\,\mathrm{d}y$$

$$+\sum_{j=1}\frac{h_{yj}}{h_0}\int_0^{l_x}\int_0^{l_y} D(y-y_j)f_{xm}f_{x\bar{m}}f_{yn}f_{y\bar{n}}\,\mathrm{d}x\,\mathrm{d}y$$

$$+\sum_{i=1}\sum_{j=1}\frac{h_{xyi,j}-h_{xi}-h_{yj}}{h_0}\int_0^{l_x}\int_0^{l_y} D(x-x_i)D(y-y_j)f_{xm}f_{x\bar{m}}f_{yn}f_{y\bar{n}}\,\mathrm{d}x\,\mathrm{d}y. \tag{16}$$

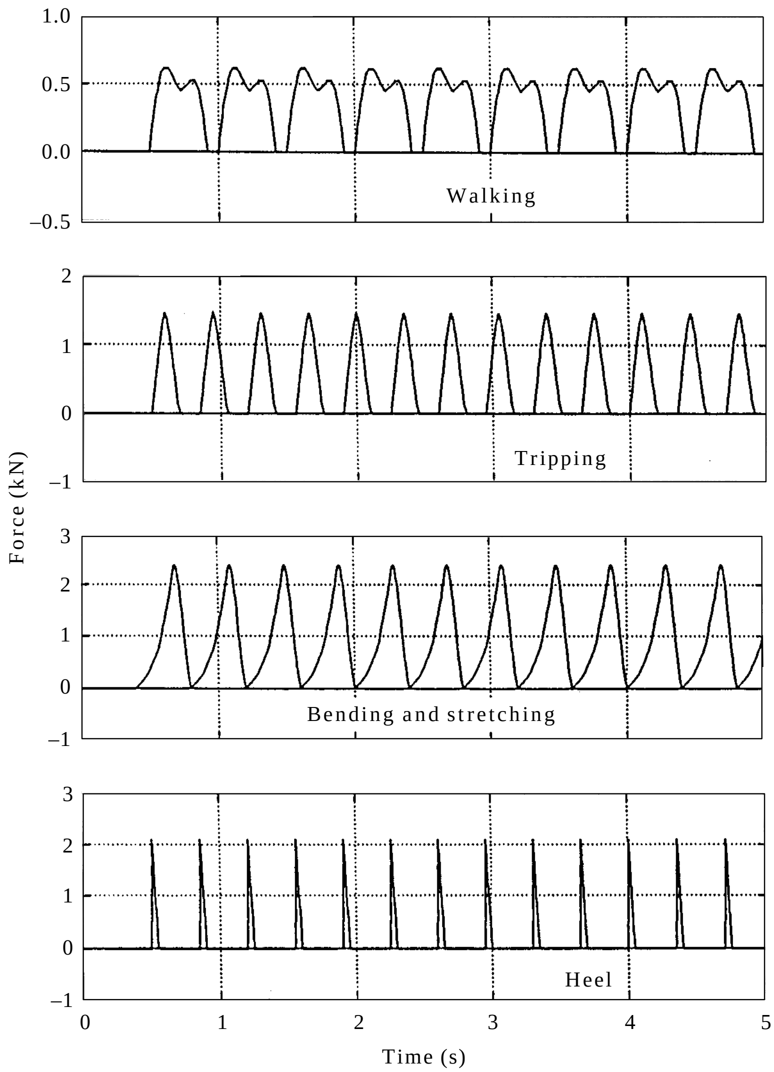

Figure 3. Test loads for building slab.

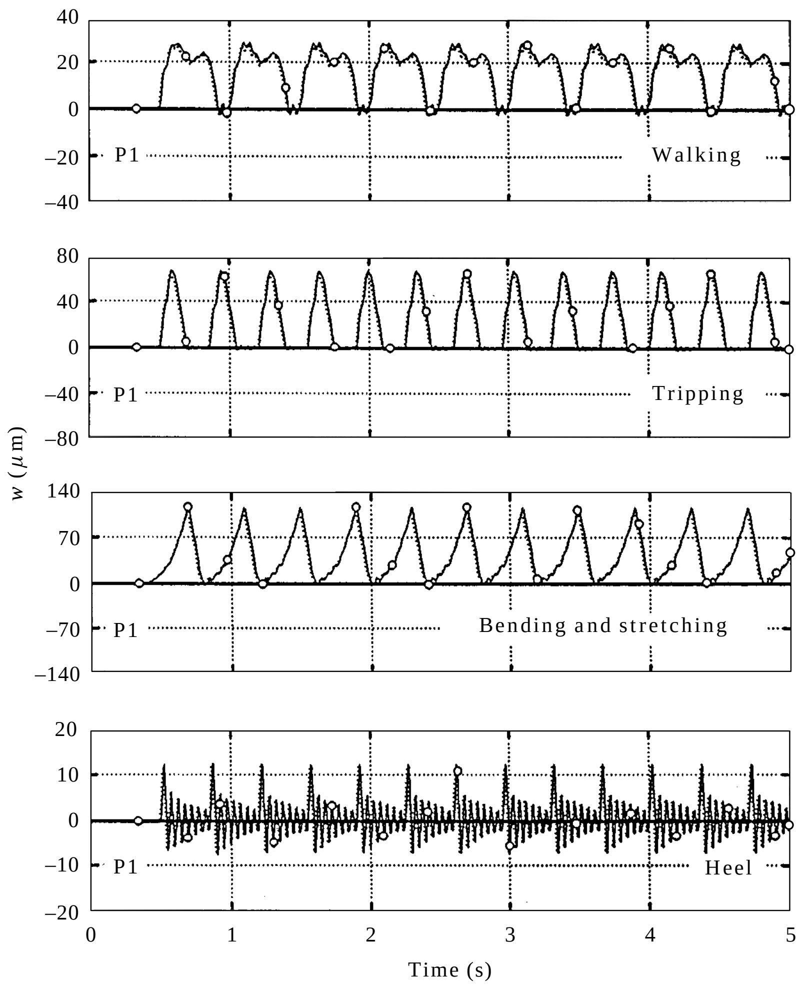

Figure 4. Dynamic deflections for a clamped plate P1 subjected to an unmoving load:—, Wilson-θ method;----, approximate solution; –○–, FEM.

The use of the characteristic function proposed here is powerful in the integral calculation including the characteristic function. Namely, for a function $f(x)$, the integral calculation including the characteristic function $D(x - x_i)$ can be written as

$$\int_0^{l_x} D(x - x_i)f(x)\,\mathrm{d}x = \int_{x_i - (b_{xi}/2)}^{x_i + (b_{xi}/2)} \left[\int_0^{l_x} \delta(x - \xi)f(x)\,\mathrm{d}x\right]\mathrm{d}\xi = \int_{x_i - (b_{xi}/2)}^{x_i + (b_{xi}/2)} f(\xi)\,\mathrm{d}\xi, \quad (17)$$

in which $\delta(x - \xi)$ is the Dirac function and ξ is a supplementary variable of x. The nth derivatives of the characteristic functions can therefore be expressed as

$$\int_0^{l_x} D^{(n)}(x - x_i)f(x)\,\mathrm{d}x = \int_{x_i - (b_{xi}/2)}^{x_i + (b_{xi}/2)} (-1)^n f^{(n)}(\xi)\,\mathrm{d}\xi, \quad (18)$$

in which the superscripts enclosed within parentheses indicate the differential order. The reduced integral is easily calculated by means of Chebyshev's formula.

TABLE 1

Maximum dynamic deflections (μm) of clamped plates with stepped thickness subjected to an unmoving load

Type	Load	Wilson-θ method	Approximate	FEM
P1	Walking	28·99 (1·02)	27·39 (0·97)	28·51
	Tripping	66·71 (1·04)	62·97 (0·99)	64·14
	Bending and Stretching	112·0 (0·97)	105·7 (0·91)	116·1
	Heel	125·0 (1·07)	116·4 (1·02)	119·4
P4	Walking	20·11 (1·09)	16·94 (0·92)	18·57
	Tripping	46·24 (0·98)	38·96 (0·83)	47·47
	Bending and Stretching	75·49 (1·09)	63·45 (0·91)	69·69
	Heel	84·69 (0·99)	73·33 (0·86)	85·58

Note: (Maximum deflection ratio) = present theory/FEM.

4. APPROXIMATE SOLUTION EXCLUDING THE EFFECT OF ADDITIONAL MASS DUE TO MOVING LOADS

Equation (11) is a coupled equation with the variable coefficients because of the effect of additional mass due to moving loads. Then, in order to present an approximate solution as closed-form for practical use, consider the following assumptions: (1) the effect of additional mass due to moving loads is negligible, and (2) only the diagonal terms in $K_{\bar{m}\bar{n}mn}$ are considered. Thus, equation (11) is approximated as the following uncoupled equation:

$$\delta\Phi_{mn}: K_{mnmn}[\ddot{\Phi}_{mn}(t) + 2h_{mn}\omega_{mn}\dot{\Phi}_{mn}(t)] = \frac{1}{m_0} Q_{mn}(t). \qquad (19)$$

Equation (19) agrees with equation (43) proposed by Takabatake *et al*. [8]. The problem of dynamic behaviour for current plates subjected to moving loads is an interesting subject

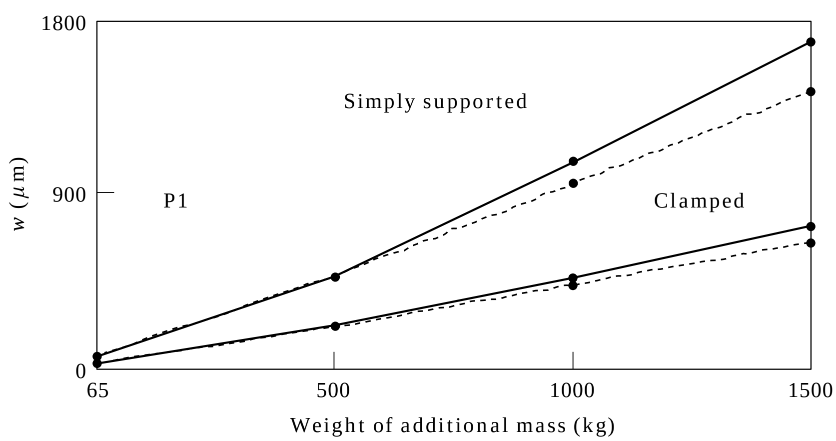

Figure 5. The effect of additional mass due to moving loads for plate P1 subjected to walking: —, includes the effect of additional mass; ----, excludes the effect of additional mass.

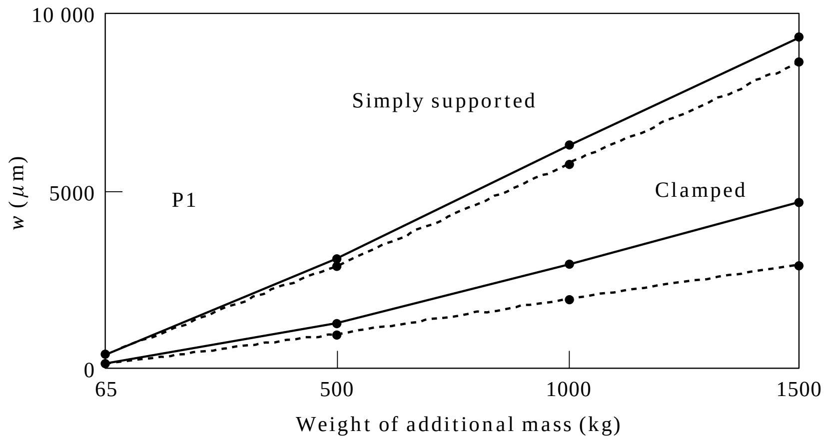

Figure 6. The effect of additional mass due to moving loads for plate P1 subjected to heel: —, includes the effect of additional mass; ----, excludes the effect of additional mass.

in practice. So, in this section it is valuable to present explicitly the analytical method of the plates subjected to moving loads. Then, the general solution of equation (19) is

$$\Phi_{mn}(t) = \exp(-h_{mn}\omega_{mn}t)[C_1 \sin \omega_{Dmn}t + C_2 \cos \omega_{Dmn}t] + \frac{1}{K_{mnmn}m_0\omega_{Dmn}}\int_0^t \exp[-h_{mn}\omega_{mn}(t-\tau)] \sin \omega_{Dmn}(t-\tau)Q_{mn}(\tau)\,d\tau, \qquad (20)$$

in which C_1 and C_2 are constants determined from the initial conditions, ω_{mn} is the natural frequency of current plate, and $\omega_{Dmn} = \omega_{mn}\sqrt{1-h_{mn}^2}$. The Duhamel integral in equation (20) may be calculated approximately as follows:

$$w_{mn}(t) = A_{mn}(t) \sin \omega_{Dmn}t - B_{mn}(t) \cos \omega_{Dmn}t, \qquad (21)$$

in which $A_{mn}(t)$ and $B_{mn}(t)$ are defined by

$$A_{mn}(t) = \frac{1}{K_{mnmn}m_0\omega_{Dmn}}\int_0^t \exp[-h_{mn}\omega_{mn}(t-\tau)] \cos \omega_{Dmn}\tau Q_{mn}(\tau)\,d\tau, \qquad (22)$$

$$B_{mn}(t) = \frac{1}{K_{mnmn}m_0\omega_{Dmn}}\int_0^t \exp[-h_{mn}\omega_{mn}(t-\tau)] \sin \omega_{Dmn}\tau Q_{mn}(\tau)\,d\tau. \qquad (23)$$

$A_{mn}(t)$ are expressed approximately with the incremental form as

$$A_{mn}(t) \approx A_{mn}(t-\Delta t)\exp(-h_{mn}\omega_{mn}\Delta t) + \frac{1}{K_{mnmn}m_0\omega_{Dmn}}\int_{t-\Delta t}^t \exp[-h_{mn}\omega_{mn}(t-\tau)] \cos\omega_{D_{mn}}\tau Q_{mn}(\tau)\,dt. \qquad (24)$$

in which the first term on the right side indicates the value of A_{mn} at time $t-\Delta t$ and Δt is incremental time. Similarly,

$$B_{mn}(t) \approx B_{mn}(t-\Delta t)\exp(-h_{mn}\omega_{mn}\Delta t) + \frac{1}{K_{mnmn}m_0\omega_{Dmn}}\int_{t-\Delta t}^t \exp[-h_{mn}\omega_{mn}(t-\tau)] \sin\omega_{D_{mn}}\tau Q_{mn}(\tau)\,d\tau. \qquad (25)$$

Consider a concentrated load $p(x, y, t)$ advancing along from the left support to the right one at $y = \eta$ with constant velocity v_0.
Then

$$p(x, y, t) = \delta(x - v_0 t)\delta(y - \eta)\bar{p}(t), \tag{26}$$

in which $\bar{p}(t)$ is the amplitude of the moving load, depending on time.

The substitution of equation (26) into equation (15) becomes

$$Q_{\bar{m}\bar{n}}(t) = f_{\bar{n}}(y - \eta) f_{\bar{m}}(v_0 t)\bar{p}(t). \tag{27}$$

5. NUMERICAL RESULTS

To examine the solution proposed here for an isotropic rectangular plate with eccentrically-stepped thickness, numerical computations were carried out for four kinds of simply supported and clamped rectangular plates with eccentrically-stepped thickness, as shown in Figure 2. The data used are as follows: span length $l_x = l_y = 6$ m; the slab's thickness $h_0 = 0{\cdot}12$ m; Young's modulus $E = 2{\cdot}06 \times 10^{10}$ Pa $(2{\cdot}1 \times 10^5$ kgf/cm^2,

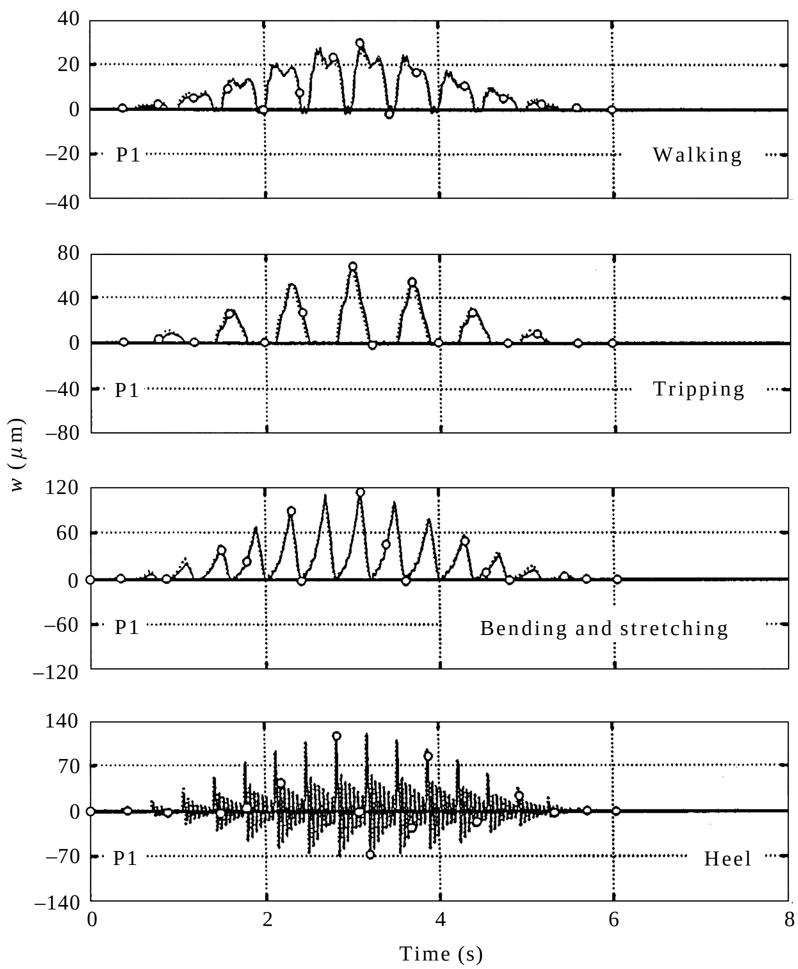

Figure 7. Dynamic deflections for a clamped plate P1 subjected to a moving load: —, Wilson-θ method; ----, approximate solution; —○—, FEM.

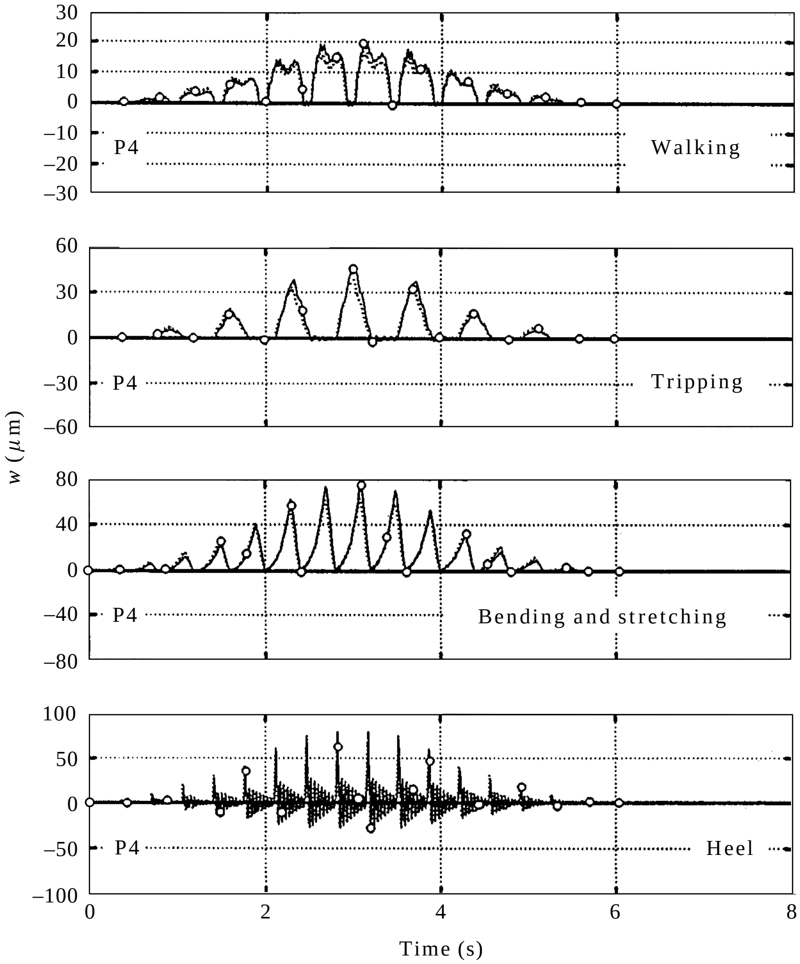

Figure 8. Dynamic deflections for a clamped plate P4 subjected to a moving load:—, Wilson-θ method;----, approximate solution; —○—, FEM.

$2{\cdot}987 \times 10^6$ psi); Poisson's ratio $\nu = 0{\cdot}17$; and mass density $\rho = 2402$ N s^2/m^4 (244·9kgf s^2/m^4, 4·660 lb s^2/ft^4); the variation of the stepped thickness $h_i/h_0 = 0.5$ and $h_j h_0 = 0{\cdot}5$. The damping constants, h_{mn}, are assumed to be 0·03 for all modes.

The external lateral loads take test loads checking the disturbing effect of the vibration of floors caused by people walking and other everyday usage, as shown in Figure 3. The test loads consist of four loading types: walking, tripping, bending and stretching, and heel. These loading types are produced for 65 kg weight and the velocity for moving loads is $v_0 = 1$ m/s except for $v_0 = 2$ m/s for tripping. The current shape functions f_{xm} and f_{yn} take the following well-known natural functions for the simply supported beam and clamped beam, respectively,

$$f_{xm} = \sin \frac{m\pi x}{l_x} \tag{28}$$

$$f_{xm} = \cosh\left(\frac{\lambda_m x}{l_x}\right) - \cos\left(\frac{\lambda_m x}{l_x}\right) - \frac{\cosh(\lambda_m) - \cos(\lambda_m)}{\sinh(\lambda_m) - \sin(\lambda_m)}\left[\sinh\left(\frac{\lambda_m x}{l_x}\right) - \sin\left(\frac{\lambda_m x}{l_x}\right)\right], \tag{29}$$

in which λ_m represents well known constants.

TABLE 2

Maximum dynamic deflections (μm) of clamped plates with stepped thickness subjected to a moving load

Type	Load	Wilson-θ method	Approximate	FEM
P1	Walking	28·73 (0·98)	27·19 (0·93)	29·49
	Tripping	65·05 (0·99)	62·60 (0·95)	65·88
	Bending and Stretching	111·2 (0·98)	105·1 (0·92)	114·2
	Heel	121·1 (1·02)	116·4 (0·98)	119·4
P4	Walking	20·06 (1·00)	16·88 (0·84)	20·09
	Tripping	46·82 (1·00)	38·58 (0·82)	47·04
	Bending and Stretching	74·67 (0·99)	62·94 (0·84)	75·66
	Heel	79·27 (1·04)	68·19 (0·90)	76·23

Note: (Maximum deflection ratio) = present theory/FEM.

Firstly, Figure 4 shows the dynamic deflections at the midspan of the clamped plates of type P1, subjected to the unmoving test loads at the midspan. In this figure, the solid lines indicate values obtained from the numerical computation using the Wilson-θ method, the broken lines indicate values obtained from the approximate solution, and the solid lines with circles indicate values obtained from the FEM code NASTRAN, in which the finite element used is an isotropic rectangular plate element with 20×20 divisions for the whole plate. The difference between solid lines and broken lines is too small to plot. Table 1 shows the maximum dynamic deflections and the maximum deflection ratios obtained from the proposed method compared to values obtained from FEM. The numerical results show that the results obtained from the proposed method are in relatively good agreement with the results obtained from FEM.

Secondly, Figures 5 and 6 show the effect of moving additional mass due to moving loads, on the dynamic deflections at the midspan of type P1, subjected to various moving loads of type walking and heel, respectively. In these figures the axis of abscissa indicates the variation of weight of the additional mass. These results are obtained from solving equation (11) by means of the Wilson-θ method. The moving loads move along $\eta = l_y/2$ in equation (26) and $v_0 = 1$ m/s except for $v_0 = 2$ m/s for tripping. It follows from these figures that the effect of additional mass due to moving loads increases significantly as the additional mass due to moving loads becomes heavier. The effect of additional mass due to moving loads must be considered for plates subjected to heavy-weight moving loads, such as the motorcar and airplane. However, the effect is negligible on the vibration of building floors caused by people walking and other everyday usage.

Thirdly, in order to present practical uses for building slabs, the effect of additional mass due to moving loads is neglected in the following numerical computations. Figures 7 and 8 show the dynamic deflections, excluding the effect of moving additional mass, at the midspan of the clamped plates of types P1 and P4, subjected to moving test loads, in which $\eta = l_y/2$ in equation (26) and $v_0 = 1$ m/s except for $v_0 = 2$ m/s for tripping. Table 2 shows the maximum dynamic deflections and the maximum deflection ratios. The numerical results show that the analytical method proposed here is also applicable to the dynamic analyses of current plates.

For these numerical models the series in the theory proposed here converges very rapidly. The consideration of each of the nine terms for m and n gives sufficient accuracy for all practical purposes.

6. CONCLUSION

The general analysis methods for an isotropic rectangular plate with arbitrarily and eccentrically-stepped thickness, subjected to moving loads along with the effect of additional mass, have been presented by extending the theory proposed by Takabatake *et al*. [8]. The effect of additional mass due to moving loads becomes significant when the amplitude of additional mass is heavy-weight. However, the effect has been clarified to be negligible on the the vibration of floors caused by people walking and other every usage.

Also, the approximate but accurate solutions proposed here, excluding the effect of additional mass due to moving loads, have been clarified numerically to be usable in the preliminary stage. Therefore, by comparing the dynamic response obtained from the proposed theory with criteria and assessment for human response, such as the sensitive curve of Meister [15], International Standard ISO 6897, and other standards in each country, trouble in building slabs is avoidable in the preliminary stage of the design. Kushida [16] discussed the relationships among these criteria and guidelines for evaluation of habitability to building vibrations. Then, further research based on design sensitivity analysis will be necessary to suggest the optimum rigidity and mass for slabs which cannot satisfy the requirement on building vibration.

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APPENDIX: NOTATION

b_{xi}, b_{yj}, h_{xi}, h_{yj}	i and jth stepped widths and heights, respectively
c	damping coefficients
D_0	flexural rigidity of solid plate
$d(x, y)$	coefficient of flexural rigidity
$D(x - x_i)$, $D(y - y_j)$	characteristic functions
E, v	Young's modulus and Poisson's ratio
f_{xm}, f_{yn}	shape functions
h_0	reference slab height
h_{mn}	damping constants
$K_{\bar{m}\bar{n}mn}$, $K^{(1)}_{\bar{m}\bar{n}mn}$, $K^{(2)}_{\bar{m}\bar{n}mn}$	coefficients
m_0	mass per unit area of solid plate
$\bar{m}$	additional mass due to moving load
p	external lateral loads
$Q_{\bar{m}\bar{n}}$	load term
v_x, v_y	Kirchhoff's supplementary forces
w	deflection
α_h	coefficient of mass
$\delta(x - x_i)$, $\delta(y - y_j)$	Dirac functions
Δt	incremental time
ρ	mass density
ω_{mn}	natural frequency
ω_{Dmn}	natural frequencies of damped plate